Fokker-Planck equation for nonequilibrium competing dynamic models

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Nonequilibrium lattice models are characterized by the dependence of their stationary properties on the details of their underlying microscopic dynamic. Any mesoscopic description of these systems, such as a Fokker-Planck equation (FPE), should reflect this microscopic dependence. In this paper we construct a FPE from a given master equation whose dynamics is a superposition of rates, each one alone driving the system to a different equilibrium state. The FPE obtained exactly reproduces (i) locally the exact stationary nonequilibrium distribution around its extremals and (ii) globally the equilibrium stationary distribution when only one of the rates is acting. In any case the FPE depends on the analytical form of the microscopic rates.

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I. INTRODUCTION

The behavior of the brain, pattern formation in liquids, and the prediction of earthquakes form part of a set of problems associated with so-called *complex systems*. A complex system is usually a system with an infinite number of degrees of freedom with a well defined microscopic dynamics, but from which we are unable *a priori* to predict or to describe the system's macroscopic behavior. The study of these systems is a growing field in which several scientific disciplines converge in an attempt to extract some general law about the behavior of these systems and/or to find some systematic method to characterize and classify them.

There are two main approaches in the theoretical study of complex systems. They are based on a microscopic master equation and on a mesoscopic type of equation modelings, respectively.

A. Microscopic master equation modeling

The system is modeled by using a set of continuous or discrete variables defined over the real space. These variables follow a local Markovian dynamics approximating the underlying real microscopic one of the complex system in question. The system is theoretically described by a microscopic master equation which is equivalent to an infinite set of time differential equations governing the behavior of all possible system correlations. These equations are usually strongly coupled to each other and it is far from trivial to solve them. Therefore, it is common to use, for example, mean field types of approximations, perturbation schemes, and/or Monte Carlo computer simulations in order to get some picture of how the system behaves. Due to the absence of a well developed theory (similar to Gibbs ensembles for equilibrium problems) to deal with this class of systems and the lack of defined limits of validity of the schemes used, the results obtained are usually controversial (see, for example, Refs. [1,2]) and/or qualitatively inconsistent when using different approaches (see, for example, Refs. [3–5]). Because of the unreliable nature of the analytical approximations, it is usually necessary to perform a Monte Carlo computer experiment to *see directly* the real behavior of the model. The main advantage in working at this level is that it is possible to control the microscopic dynamic details and study their influence on the macroscopic behavior. We should mention that some relevant exact results have been obtained for this class of models, for example, the dynamical properties for the one-dimensional Ising model with a specific dynamics [6], the stationary properties of nonequilibrium systems with a mean field type of dynamics (see, for instance, Ref. [3]), the rigorous construction of a continuous reaction-diffusion equation from a particular microscopic stochastic model [7], and the existence of effective Hamiltonians which describe the stationary states of some nonequilibrium models [8].

B. Mesoscopic descriptions

The system's dynamical behavior is described through a finite set of mesoscopic differential equations whose structure captures some specific system symmetries, conservation laws, and/or any other *macroscopic* properties. Some well-known relevant examples include the Boltzmann equation, the Fokker-Planck equation, and the Navier-Stokes equation (see, for example, [9]). The connection between these equations and the microscopic system that they attempt to describe is, in most of the cases, intuitive and nonrigorous, but it appears to be useful in order to know the corresponding limits of application. Within these limits, it is remarkable how good the macroscopic descriptions that we obtain from them are. All these equations have something in common: they describe a *dynamical averaged* system behavior which is the result of space and time rescaling. Even though the

aforementioned equations are far from trivial, at least there exist systematic theoretical tools to extract some valid information from them (see, for instance, Refs. [9–11]). There is, however, a problem in applying this kind of equation to the study of complex systems: any direct dependence on the microscopic dynamical details is lost or not accurately contemplated. Therefore, for *each* phenomena observed, it is necessary to build, *a posteriori*, the corresponding mesoscopic equation which describes it.

We see from the above two points of view that efforts to connect the master equation type of descriptions with mesoscopic equations are interesting by themselves when dealing with complex systems. In this paper we explain a systematic way to connect both descriptions above for a particular family of nonequilibrium competing dynamic models [12], that is, models whose microscopic dynamic is the superposition of several ones which, acting alone, drive the system to different equilibrium states. From a microscopic Markovian master equation, which defines the model, we would like to construct a mesoscopic Fokker-Planck equation (FPE) with four minimum desired properties.

(i) The FPE's should depend explicitly on the microscopic rates (as it is observed in Monte Carlo computer experiments of closely related nonequilibrium lattice models [13]).

(ii) The dynamics associated with the deterministic part of the equation should be described exactly.

(iii) Its stationary distribution should be the corresponding equilibrium Gibbsian one (independent on the microscopic rate used), when a unique microscopic mechanism acts.

(iv) In general, its stationary nonequilibrium distribution should be at least locally exact around the extremals of the true one.

None of the FPE's commonly used in the literature (say, for instance, Hohenberg and Halperin's model A [14] or the Kramers-Moyal FPE's [10]), fulfills at the same time these four requirements.

In order to get the desired FPE we follow a two-step strategy by answering affirmatively the following two questions: (1) Is it possible, on physical grounds, to get a rate-dependent mesoscopic description from a Markovian master equation? (2) Is it possible to find a second-order partial differential equation with the properties (iii) and (iv). The first answer is affirmative because, as we will see, there exists a natural parameter on these models Ω, which characterizes the coarse graining from a microscopic lattice model to the continuous equivalent one. This parameter is used to expand the master equation. From this expansion we get a set of equations whose solution should give us the exact nonequilibrium distribution order by order in Ω. In particular, from it we obtain a rate-dependent deterministic equation when $\Omega \to \infty$.

The answer to the second question is partially affirmative. That is, we manage to adjust our FPE in such a form that it describes the first two leading orders in the Ω parameter expansion of the exact nonequilibrium stationary distribution.

II. MODEL

Let us define first an equilibrium model which will be the basis of our nonequilibrium competing dynamic models. On a d-dimensional cubic lattice we allocate at each site $\underline{x} \in Z^d$ a spin variable, say $s(\underline{x})=\pm 1$. The system interaction Hamiltonian is given by $H_0(\underline{s};\underline{K})$, where $\underline{s}=\{s(\underline{x})|\underline{x} \in Z^d\}$ and $\underline{K}=\{K_i\}_{i=1}^n$ is the set of parameters defining the interaction. Because the spin variables are discrete and our goal is to construct a continuous type of equation, we define a new set of variables

$$\Phi_\Omega(\underline{x}) \equiv \frac{1}{\Omega} \sum_{\underline{y} \in \Lambda_{\underline{x}}^\Omega} s(\underline{y}) , \quad \underline{x} \in Z^d , \tag{2.1}$$

where $\Lambda_{\underline{x}}^\Omega \subset Z^d$ is a volume centered at $\underline{x}$ containing Ω lattice sites. Note that the Φ_Ω variables may take the values $-1+2l/\Omega$, with $l=0,1,\ldots,\Omega$. We assume that the equivalent interaction Hamiltonian H^Ω is such that it can be written in these new variables as

$$H^\Omega(\underline{\Phi}_\Omega;\underline{K}) = \sum_{\underline{x} \in Z^d} h^\Omega(\underline{\Phi}_\Omega(\underline{x});\underline{K}) , \tag{2.2}$$

where $\underline{\Phi}_\Omega(\underline{x})$ is some generic finite local set of Φ_Ω variables around $\underline{x}$ and $\underline{\Phi}_\Omega=\{\Phi_\Omega(\underline{x}),\underline{x} \in Z^d\}$. An example of a one-dimensional Hamiltonian is $H^\Omega(\underline{\Phi}_\Omega;\{\mu,\lambda\})$ $=\sum_x (\{\Omega[\Phi_\Omega(x) - \Phi_\Omega(x-1)]\}^2 + \mu^2 \Phi_\Omega^2(x) + \lambda \Phi_\Omega^4(x))$. Let us remark here that we are not interested in how to build explicitly the Hamiltonian (2.2) from the original lattice one H_0. It could be an interesting subject by itself, but it is not a relevant point for the goals of this paper. We choose the radius of any region $\Lambda_{\underline{x}}^\Omega$ of the order of the spin system correlation length. In this case we can guarantee that $\Phi_\Omega(\underline{x})$ is a smooth function of $\underline{x}$, i.e., we can write $\Phi_\Omega(\underline{x})=\Phi(\Omega^{-1/d}\underline{x})$, where $\Phi(\underline{r})$ is assumed to be an analytic function in $\underline{r}$. If a_0 is the lattice spacing, we define a rescaled one $a = \Omega^{1/d} a_0$. The continuum limit is then defined by taking $|a_0| \to 0$ and $\Omega \to \infty$ while fixing $a = 1$. In this limit Eq. (2.2) becomes

$$H^\Omega(\underline{\Phi};\underline{K}) = \Omega \int_{R^d} d\underline{r} \, h(\underline{\Phi}(\underline{r});\underline{K}) . \tag{2.3}$$

This continuum limit implies that we can only study phenomena with a typical macroscopic length. Notice that we have considered that the original spin Hamiltonian is such that in the final continuum one does not depend on Ω^0 corrections. This situation is not the rule and, in general, Ω^0 corrections will appear. However, the spirit of the approach would not change with this inclusion and for simplicity we will not consider them.

We assume now that the above spin system has a stochastic spin flip dynamics; that is, a spin variable change its sign in a random way according to some probability function. A spin change will be reflected in a change of the order $\pm 2/\Omega$ in the fields Φ in which it is contained. Therefore, a field configuration $\underline{\Phi}$ will evolve towards its equilibrium state characterized by the set of parameters $\underline{K}$, by means of a Markov process represented by a Markovian master equation (MME). In this paper we consider that the probability at time t that the system configuration is $\underline{\Phi}$, say $P_t^\Omega(\underline{\Phi};\underline{K})$, follows the MME

$$\partial_t P_t^{\Omega}(\underline{\Phi};\underline{K}) = \int_{R^d} d\underline{r} \int_R d\eta \, f(\eta)[w^{\Omega}(\underline{\Phi}^{\eta,\underline{r}} \to \underline{\Phi};\underline{K})P_t^{\Omega}(\underline{\Phi}^{\eta,\underline{r}};\underline{K}) - w^{\Omega}(\underline{\Phi} \to \underline{\Phi}^{\eta,\underline{r}};\underline{K})P_t^{\Omega}(\underline{\Phi};\underline{K})] , \qquad (2.4)$$

where $\underline{\Phi}^{\eta,\underline{r}} = \{\Phi(\underline{r}') + \eta/\Omega\delta_{\underline{r},\underline{r}'}, \underline{r}' \in R^d\}$. The function $f(\eta)$ is the distribution of the possible field increments and it is assumed to be symmetric, i.e., $f(\eta) = f(-\eta)$, and analytic around the origin, and $w^{\Omega}(\underline{\Phi} \to \underline{\Phi}';\underline{K})$ is the probability per unit time that the system changes from the state $\underline{\Phi}$ to the state $\underline{\Phi}'$. In order to guarantee that the system's stationary state will be an equilibrium one given by the Gibbsian weight $P_{eq}^{\Omega}(\underline{\Phi};\underline{K}) \propto \exp[-H^{\Omega}(\underline{\Phi};\underline{K})]$, it is sufficient to build the rates w^{Ω} as

$$w^{\Omega}(\underline{\Phi} \to \underline{\Phi}';\underline{K}) = D(H^{\Omega}(\underline{\Phi}';\underline{K}) - H^{\Omega}(\underline{\Phi};\underline{K})) ,$$

$$D(\lambda) = e^{-\lambda}D(-\lambda) \geq 0 , \quad D(0) = 1 . \qquad (2.5)$$

Therefore, Eqs. (2.3)–(2.5) define a continuous dynamical model which, for almost any initial field configuration, will evolve to a given equilibrium state.

The above construction from the original lattice spin system may seem worthless because we could have written (2.3)–(2.5) as the model definition. But, as we will show, the Ω factor in front of the continuous Hamiltonian in Eq. (2.3) is essential in maintaining some dependence on the rate D at the mesoscopic level. Therefore, we thought it was necessary to show where it came from and its physical meaning.

From this equilibrium model it is easy to construct the family of nonequilibrium models which are going to study in this paper. This is done by linearly superposing several of the above equilibrium rates with different $\underline{K}$ parameters [12]. In this form, the system is dynamically frustrated in trying each rate to drive the system to different equilibrium states and, a priori, the stationary state is expected to be a non-Gibbsian one (i.e., there is no effective Hamiltonian describing the stationary state).

The MME corresponding to this class of nonequilibrium models is, similarly to Eq. (2.4), written as

$$\partial_t P_t^{\Omega}(\underline{\Phi}) = \int_{R^d} d\underline{r} \int_R d\eta \, f(\eta)[w^{\Omega}(\underline{\Phi}^{\eta,\underline{r}} \to \underline{\Phi})P_t^{\Omega}(\underline{\Phi}^{\eta,\underline{r}})$$

$$- w^{\Omega}(\underline{\Phi} \to \underline{\Phi}^{\eta,\underline{r}})P_t^{\Omega}(\underline{\Phi})] , \qquad (2.6)$$

with

$$w^{\Omega}(\underline{\Phi} \to \underline{\Phi}') = \int_{R^n} d\underline{K} \, p(\underline{K})w^{\Omega}(\underline{\Phi} \to \underline{\Phi}';\underline{K}) , \qquad (2.7)$$

where $p(\underline{K})$ is an a priori given distribution which is part of the model definition. Equations (2.5)–(2.7) give us the complete microscopic dynamical definition of our model. It is not obvious at all how to get some insight about the system's macroscopic behavior from these equations. The frontal undertaking of the MME is usually worthless and therefore it is necessary to find a different point of view in order to analyze this class of systems. The point of view that we have used in this paper is the construction, from the above MME, of a mesoscopic equation with some a priori minimum required properties.

We see in Eq. (2.6) that there is a natural parameter Ω which reflects a coarse-grained origin of our continuous model from a microscopic lattice spinlike model. As we commented, this parameter should be large enough in order to define correctly the continuous limit. Therefore, we have a natural parameter to formally expand the MME (2.6). The result of this expansion is the well-known Kramers-Moyal expansion (see, for instance, Refs. [10,15,16]), which in our case reads

$$\partial_t P_t^{\Omega}(\underline{\Phi}) = \int_{R^d} d\underline{r} \int_R d\eta \, f(\eta) \left[\{ \langle\langle D(-U_{\eta}[\underline{\Phi}(\underline{r});\underline{K}] - W_{\eta}^{\Omega}[\underline{\Phi}(\underline{r});\underline{K}]) \rangle\rangle \right.$$

$$- \langle\langle D(U_{\eta}[\underline{\Phi}(\underline{r});\underline{K}] + W_{\eta}^{\Omega}[\underline{\Phi}(\underline{r});\underline{K}]) \rangle\rangle \} P_t^{\Omega}(\underline{\Phi})$$

$$\left. + \sum_{l=1}^{\infty} \frac{1}{l!} \left[\frac{\eta}{\Omega}\right]^l \frac{\delta^l P_t^{\Omega}(\underline{\Phi})}{\delta\Phi(\underline{r})^l} \langle\langle D(-U_{\eta}[\underline{\Phi}(\underline{r});\underline{K}] - W_{\eta}^{\Omega}[\underline{\Phi}(\underline{r});\underline{K}]) \rangle\rangle \right] , \qquad (2.8)$$

where

$$U_{\eta}(\underline{\Phi}(\underline{r});\underline{K}) = \eta \frac{\delta\hat{H}(\underline{\Phi};\underline{K})}{\delta\Phi(\underline{r})} ,$$

$$W_{\eta}^{\Omega}(\underline{\Phi}(\underline{r});\underline{K}) = \sum_{l=1}^{\infty} \frac{1}{(l+1)!} \frac{\eta^{l+1}}{\Omega^l} \frac{\delta^{l+1}\hat{H}(\underline{\Phi};\underline{K})}{\delta\Phi(\underline{r})^{l+1}} , \qquad (2.9)$$

$$\langle\langle A \rangle\rangle \equiv \int_{R^n} d\underline{K} \, p(\underline{K})A(\underline{K}) \quad \text{and} \quad \hat{H}(\underline{\Phi};\underline{K})$$

$\equiv \Omega^{-1}H^{\Omega}(\underline{\Phi};\underline{K})$. Some useful information can be derived from Eq. (2.8) from a perturbative scheme.

Let us assume that for large enough Ω the probability

distribution can be written as

$$P_t^{\Omega}(\underline{\Phi}) = N_t^{\Omega}\exp(-V_t^{\Omega}(\underline{\Phi})) ,$$

$$V_t^{\Omega}(\underline{\Phi}) = \Omega V_{0,t}(\underline{\Phi}) + V_{1,t}(\underline{\Phi}) + O(\Omega^{-1}) , \qquad (2.10)$$

where $N_t^{\Omega} = \int d\underline{\Phi}\exp(-V_t^{\Omega}(\underline{\Phi}))$. V_t^{Ω} is called in general time-dependent nonequilibrium potential. We are going to assume that this potential has an unique limit when the time goes to infinity, i.e., $\lim_{t \to \infty} V_t^{\Omega} = V_{st}^{\Omega}$, which we call stationary nonequilibrium potential or sometimes only nonequilibrium potential [17].

Notice the difference between a *Hamiltonian* as the one written in (2.3) and a general nonequilibrium potential: the first one is a particular class of nonequilibrium potential in which it is possible to distinguish a sort of *local* interaction between the system elements. Moreover, the (time-) nonequilibrium potential is expected to be continuous but not differentiable in some small regions in the phase space (see, for instance, Refs. [18–20] and it can be shown that it is a Lyapunov function for the underlying deterministic dynamical system [21]. Anyway, we are going to work where the potential is assumed to be locally differentiable, that is, nearby its local minima.

When we substitute Eq. (2.10) into Eq. (2.8) and keep the leading orders in Ω we get

$$\partial_t \ln N_t^\Omega - \Omega \partial_t V_{0,t}(\underline{\Phi}) - \partial_t V_{1,t}(\underline{\Phi}) + O(\Omega^{-1})$$

$$= \int_{R^d} d\underline{r} \int_R d\eta \, f(\eta) \left[\langle\!\langle D(U_\eta[\underline{\Phi}(\underline{r});\underline{K}]) \rangle\!\rangle [e^{\eta[\delta V_{0,t}/\delta\Phi(\underline{r})]} - 1] \right.$$

$$\left. - \frac{\eta}{2\Omega} \frac{\delta}{\delta\Phi(\underline{r})} \{ \langle\!\langle D(U_\eta[\underline{\Phi}(\underline{r});\underline{K}]) \rangle\!\rangle (1 + e^{\eta[\delta V_{0,t}/\delta\Phi(\underline{r})]}) \} \right.$$

$$\left. + \frac{\eta}{\Omega} \langle\!\langle D(U_\eta[\underline{\Phi}(\underline{r});\underline{K}]) \rangle\!\rangle e^{\eta[\delta V_{0,t}/\delta\Phi(\underline{r})]} \frac{\delta V_{1,t}}{\delta\Phi(\underline{r})} \right] + O(\Omega^{-2}) . \tag{2.11}$$

From 2.11 we can extract some limiting results.

(i) *The deterministic equation.* When we introduce the time variable $\tau = \Omega^{-1}t$ and do the limit $\Omega \to \infty$, the solution of Eq. (2.8) reads

$$P_\tau(\underline{\Phi}) = \delta(\underline{\Phi} - \underline{v}_\tau) \tag{2.12}$$

with

$$\partial_\tau v_\tau(\underline{r}) = \int_R d\eta \, f(\eta) \eta \langle\!\langle D(U_\eta[\underline{v}_\tau(\underline{r});\underline{K}]) \rangle\!\rangle . \tag{2.13}$$

Equation (2.13) is usually called the *deterministic equation*. It is possible to show (see Appendix A) that the solutions of Eq. (2.13) are extremals of the zeroth-order nonequilibrium potential, i.e., $\delta V_{0,t}(\underline{\Phi})/\delta\Phi(\underline{r})|_{\underline{\Phi}=\underline{v}_\tau} = 0$. Let us remark here that this equation is clearly dependent on the analytical form of the microscopic rate

D and this is so because there is a Ω factor in front of the Hamiltonians (2.3) defining the rate (2.7). We may conclude that the coarse graining commented on in Sec. II is essential in order to have a correct macroscopic description (as it is shown from computer experiments [13]) and, for instance, it may deserve to be carefully studied in the future.

(ii) *The stationary state.* To study the stationary state we set the left-hand side of Eq. (2.11) equal to zero. Then, order by order in Ω, we set the right-hand side of Eq. (2.11) equal to zero and get the set of closed differential equations whose solutions are the coefficients, $V_{i,\text{st}}$, $i = 0, 1, \ldots$, of the nonequilibrium potential. In particular, the equations arising from the first two orders are

$$\int_{R^d} d\underline{r} \int_R d\eta \, f(\eta) \langle\!\langle D(U_\eta[\underline{\Phi}(\underline{r});\underline{K}]) \rangle\!\rangle [e^{\eta[\delta V_{0,\text{st}}/\delta\Phi(\underline{r})]} - 1] = 0 , \tag{2.14}$$

$$\int_{R^d} d\underline{r} \int_R d\eta \, f(\eta) \frac{\eta}{2} \left[\frac{\delta}{\delta\Phi(\underline{r})} \{ \langle\!\langle D(U_\eta[\underline{\Phi}(\underline{r});\underline{K}]) \rangle\!\rangle (1 + e^{\eta[\delta V_{0,\text{st}}/\delta\Phi(\underline{r})]}) \} \right.$$

$$\left. - 2 \langle\!\langle D(U_\eta[\underline{\Phi}(\underline{r});\underline{K}]) \rangle\!\rangle e^{\eta[d V_{0,\text{st}}/\delta\Phi(\underline{r})]} \frac{\delta V_{1,\text{st}}}{\delta\Phi(\underline{r})} \right] = 0 . \tag{2.15}$$

In order to find the nonequilibrium potential, one could be tempted, for example, to assume that the stationary distribution has the so-called *detailed balance property*, that is, Eqs. (2.14) and (2.15) hold *without* the $\int_{R^d} d\underline{r}$ term in front of them. But, even in the case in which we were able to isolate the first derivatives of $V_{i,\text{st}}$ from the set of Eqs. (2.14) and (2.15), i.e., $\delta V_{i,\text{st}}/\delta\Phi(\underline{r}) = F_i[\Phi(\underline{r}), \nabla\Phi(\underline{r}), \ldots, \nabla^m\Phi(r), \ldots]$, $i = 0, 1, \ldots$, it is only possible to integrate the latter equations for some particular functional forms of F_i

(such as, for example, the one given by $F_i = g_i[\Phi(\underline{r})] + a_i \nabla^2\Phi(\underline{r})$, where g_i and a_i are an arbitrary function and a constant, respectively). In general, this is not going to be the case and therefore, we cannot usually obtain the nonequilibrium potential directly from a detailed balance condition. Notice, however, that when a unique microscopic mechanism is acting, i.e., $p(\underline{K}) = \delta(\underline{K} - \underline{K}_0)$, the detailed balance property holds because of our original rate construction [see Eq. (2.5)] and then $V_{\text{st}}^\Omega(\underline{\Phi}) = H^\Omega(\underline{\Phi};\underline{K}_0)$.

III. HOW TO BUILD AN APPROPRIATE FOKKER-PLANCK EQUATION

Let us write a general FPE

$$\partial_t P_t^\Omega(\Phi) = \frac{1}{\Omega} \int_{R^d} dr \frac{\delta}{\delta\Phi(r)}$$

$$\times \left[\Xi_0[\Phi(r)] + \frac{1}{\Omega}\Xi_1[\Phi(r)] + \frac{1}{\Omega}\Xi_2[\Phi(r)]\frac{\delta}{\delta\Phi(r)} \right] P_t^\Omega(\Phi) . \tag{3.1}$$

We would like to fit the functionals Ξ_0, Ξ_1, and Ξ_2 in such a way that at least Eqs. (2.11)–(2.13) hold . That is, we expect to reproduce the two first leading terms in Ω of the true time nonequilibrium potential. Because our parameter Ω is going to be physically large, the latter will be a good approximation of the real system behavior.

In order to fit the Ξ functionals, let us substitute Eq. (2.10) into (3.1) and expand the latter in leading orders on Ω. The resulting equation reads

$$\partial_t \ln N_t^\Omega - \Omega \partial_t V_{0,t}(\Phi) - \partial_t V_{1,t}(\Phi) + O(\Omega^{-1})$$

$$= -\int_{R^d} dr \frac{\delta V_{0,t}}{\delta\Phi(r)} \left[\Xi_0[\Phi(r)] - \Xi_2[\Phi(r)]\frac{\delta V_{0,t}}{\delta\Phi(r)} \right]$$

$$- \frac{1}{\Omega}\int_{R^d} dr \left[-\frac{\delta\Xi_0[\Phi(r)]}{\delta\Phi(r)} + \Xi_0[\Phi(r)]\frac{\delta V_{1,t}}{\delta\Phi(r)} + \Xi_1[\Phi(r)]\frac{\delta V_{0,t}}{\delta\Phi(r)} + \frac{\delta\Xi_2}{\delta\Phi(r)}\frac{\delta V_{0,t}}{\delta\Phi(r)} \right.$$

$$\left. + \Xi_2[\Phi(r)]\frac{\delta^2 V_{0,t}}{\delta\Phi(r)^2} - 2\Xi_2[\Phi(r)]\frac{\delta V_{0,t}}{\delta\Phi(r)}\frac{\delta V_{1,t}}{\delta\Phi(r)} \right] + O(\Omega^{-2}) . \tag{3.2}$$

It would be a sufficient condition to reproduce an identical dynamical behavior of the time nonequilibrium potential in its first two leading orders in Ω, with the coefficients being, by direct comparison of Eq. (3.2) with Eqs. (2.11)–(2.13),

$$\Xi_0^{\text{exact}}[\Phi(r)] = -\int_R d\eta\, f(\eta)\eta \langle\!\langle D(U_\eta[\Phi(r);K]) \rangle\!\rangle$$

$$\Xi_1^{\text{exact}}[\Phi(r)] = \left[\frac{\delta V_{0,t}}{\delta\Phi(r)}\right]^{-1} \frac{\delta V_{1,t}}{\delta\Phi(r)} \int_R d\eta\, f(\eta) \langle\!\langle D(U_\eta[\Phi(r);K]) \rangle\!\rangle$$

$$\times \left\{ e^{\eta[\delta V_{0,t}/\delta\Phi(r)]}\left[-\eta + 2\left[\frac{\delta V_{0,t}}{\delta\Phi(r)}\right]^{-1}\right] - 2\left[\frac{\delta V_{0,t}}{\delta\Phi(r)}\right]^{-1} - \eta \right\}$$

$$- \frac{1}{2}\left[\frac{\delta V_{0,t}}{\delta\Phi(r)}\right]^{-1}\frac{\delta}{\delta\Phi(r)}\int_R d\eta\, f(\eta)\langle\!\langle D(U_\eta[\Phi(r);K]) \rangle\!\rangle$$

$$\times \left\{ e^{\eta[\delta V_{0,t}/\delta\Phi(r)]}\left[-\eta + 2\left[\frac{\delta V_{0,t}}{\delta\Phi(r)}\right]^{-1}\right] - 2\left[\frac{\delta V_{0,t}}{\delta\Phi(r)}\right]^{-1} - \eta \right\} , \tag{3.3}$$

$$\Xi_2^{\text{exact}}[\Phi(r)] = \left[\frac{\delta V_{0,t}}{\delta\Phi(r)}\right]^{-2}\int_R d\eta\, f(\eta)\langle\!\langle D(U_\eta[\Phi(r);K]) \rangle\!\rangle \left[e^{\eta[\delta V_{0,t}/\delta\Phi(r)]} - 1 - \eta\frac{\delta V_{0,t}}{\delta\Phi(r)} \right] ,$$

where Ξ_0^{exact} is fixed by expecting the FPE (3.1) to reproduce the exact deterministic equation (2.13). As we see, the coefficients Ξ_1^{exact} and Ξ_2^{exact} given by (3.3) depend on the explicit form of the *unknown* time nonequilibrium potential functional first derivative. If we are only interested in making sure that the exact nonequilibrium potential is the stationary solution of the FPE (3.1), then it is enough to replace $V_{i,t}$, $i = 0,1$, by $V_{i,\text{st}}$, $i = 0,1$, in Eq. (3.3). Then it is possible to construct a FPE such that its solution is the exact one up to certain order in perturbation theory? The unique answer seems to be *yes, if we know a priori the exact solution.* That is the case when dealing with systems at equilibrium where the stationary solution is the Gibbsian one. In fact, from Eqs. (3.3), it is straightforward to show the following theorem.

Theorem 1. Let $H(\Phi) = \Omega\hat{H}(\Phi)$ be the interaction Hamiltonian that characterizes the system equilibrium stationary state. Then the FPE (3.1) with coefficients

$$\Xi_0[\Phi(r)] = \Xi_0^{\text{eq}}[\Phi(r)] ,$$

$$\Xi_1[\Phi(r)] = 0 , \tag{3.4}$$

$$\Xi_2[\Phi(r)] = \frac{\Xi_0^{\text{eq}}[\Phi(r)]}{\dfrac{\delta\hat{H}(\Phi)}{\delta\Phi(r)}} ,$$

where

$$\Xi_0^{\text{eq}}[\Phi(r)] = -\int_R d\eta\, f(\eta)\eta D\left[\eta\frac{\delta\hat{H}(\Phi)}{\delta\Phi(r)}\right] , \tag{3.5}$$

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has the following properties: (i) It reproduces the exact deterministic dynamics given by Eq. (2.12) with

$$\partial_\tau p_\tau(\underline{r}) = \int_R d\eta \, f(\eta) \eta D \left[\eta \frac{\delta \hat{H}(\underline{v}_\tau)}{\delta v_\tau(\underline{r})} \right] \qquad (3.6)$$

(ii) and its stationary solution is the exact Gibbsian one $P_{st}(\underline{\Phi}) \propto \exp[-H(\underline{\Phi})]$.

Proof. (i) In the $\Omega \to \infty$ limit, the FPE coefficient Ξ_0 defines completely the deterministic equation. Therefore, we only need to show that the Ξ_0 coefficient of the FPE (3.1), given by Eqs. (3.4) and (3.5), coincides with the exact one Ξ_0^{exact}. This is, by construction, the case. (ii) This point is shown by direct substitution of the Gibbsian weight into the stationary FPE (3.1) $\partial_t P_t^\Omega = 0$.

Notice how the rate analytical form D appears explicitly in the FPE and it characterizes the system dynamical properties. This theorem was already used by Hanggi *et al.* [22,23] in order to study transient phenomena in systems with one variable for which it is always possible to find the exact stationary distribution.

When dealing with the competing dynamical system, we do not know the nonequilibrium potential and then we cannot find *a priori* any FPE which reproduces globally that potential. Anyway, we have found that, *without knowing explicitly that solution,* it is possible to construct a FPE such that its stationary solution is almost exact near the nonequilibrium potential extremals. Let us demonstrate the later assertion by means of the next two theorems.

Theorem 2. Let any FPE (3.1) be such that its coefficients have the properties

$$\Xi_0[\underline{\Phi}(\underline{r})] = \Xi_0^{exact}[\underline{\Phi}(\underline{r})] \, ,$$
$$\Xi_2[\underline{v}^*(\underline{r})] = \tfrac{1}{2} \int_R d\eta \, f(\eta) \langle\langle D(U_\eta[\underline{v}^*;\underline{K}]) \rangle\rangle \eta^2 \, , \qquad (3.7)$$

where $\underline{v}^*$ is any dynamically stable solution for the stationary deterministic equation (2.13), i.e., $\partial_\tau \underline{v}^*(\underline{r}) = 0$. Then (i) it reproduces the exact deterministic dynamics given by Eqs. (2.12) and (2.13) and (ii) the $V_{0,st}$ part of its stationary solution almost coincides with the exact one in a suitable neighborhood of $\underline{v}^*$.

Proof. (i) Same as (i) in Theorem 1. (ii) The component $V_{0,st}^{exact}$ of the exact stationary nonequilibrium potential is the solution of the Hamilton-Jacobi type of Eq. (2.14):

$$H_0^{exact}(\underline{\pi}, \underline{\Phi}) \equiv \int_R d\underline{r} \int_R d\eta \, f(\eta) \langle\langle D(U_\eta[\underline{\Phi}(\underline{r}); \underline{K}]) \rangle\rangle$$
$$\times [e^{\eta\pi(\underline{r})} - 1] = 0 \, , \qquad (3.8)$$

where $\pi(\underline{r}) = \delta V_{0,st}^{exact}/\delta \Phi(\underline{r})$. Similarly, from Eq. (3.2), we see that the component $V_{0,st}^{FP}$ of the stationary nonequilibrium potential of a FPE (3.1) is given by

$$H_0^{FP}(\underline{\pi}', \underline{\Phi}) \equiv -\int_R d\underline{r} \, \pi'(\underline{r}) \{\Xi_0[\underline{\Phi}(\underline{r})] - \Xi_2[\underline{\Phi}(\underline{r})]\pi'(\underline{r})\}$$
$$= 0 \, , \qquad (3.9)$$

where $\pi'(\underline{r}) = \delta V_{0,st}^{FP}/\delta \Phi(\underline{r})$. Both nonequilibrium potentials have the same minima because both of them produce the same deterministic equation (see Appendix A).

Let us assume that these nonequilibrium potentials are differentiable in a suitable small region around any of its

local minima. This is a reasonable conjecture because it has been shown that this potential is almost everywhere differentiable (see, for instance, Refs. [21,24] and references therein). Therefore, in such a region we can apply the theorem in Appendix B, which relates a Hamiltonian-Jacobi type of equation with a Hamiltonian dynamical system. Notice that the latter theorem cannot be applied directly to Eq. (3.8) or (3.9) because we cannot guarantee the C^∞ character of the potential in the full configuration space.

To build the corresponding potentials locally around of their minima, we should solve first the dynamical systems characterized by the Hamiltonians $H_0^{exact}(\underline{\pi}, \underline{\Phi})$ and $H_0^{FP}(\underline{\pi}, \underline{\Phi})$, where $\underline{\pi}$ symbolizes now the conjugate moments of $\underline{\Phi}$, i.e.,

$$\dot{\Phi}(\underline{r}) = \frac{\delta H_0(\underline{\pi}, \underline{\Phi})}{\delta \pi(\underline{r})} \, , \quad \dot{\pi}(\underline{r}) = -\frac{\delta H_0(\underline{\pi}, \underline{\Phi})}{\delta \Phi(\underline{r})} \, . \qquad (3.10)$$

The trajectories associated with these dynamical systems have to cross the point $M = (\underline{0}, \underline{v}^*)$.

Finally, the potential at point $\underline{\Phi}$ is equal to the integral of π along the trajectory connecting the point $(\underline{\pi}_0, \underline{\Phi})$ with M, where π_0 is fixed by choosing the trajectory that reaches M when $t \to -\infty$ [18–20,25].

Therefore, we only need to show that the Hamiltonians H_0^{exact} and H_0^{FP} are almost equal in a neighborhood of a common minimum in order to show that the two sets of partial differential equations have an almost identical solution around it. In our case, the local expansion of the Hamiltonians (3.8) and (3.9) around the minimum $(\underline{0}, \underline{v}^*)$ is

$$H_0^{exact}(\underline{\pi}, \underline{\Phi})$$
$$= -\int_{R^d \otimes R^d} d\underline{r} \, d\underline{r}' \Upsilon(\underline{r}') \pi(\underline{r}) \frac{\delta \Xi_0^{exact}[v^*(\underline{r})]}{\delta v^*(\underline{r}')}$$
$$+ \tfrac{1}{2} \int_{R^d} d\underline{r} \, \pi(\underline{r})^2 \int_R d\eta \, f(\eta) \eta^2 \langle\langle D(U_\eta[\underline{v}^*; \underline{K}]) \rangle\rangle$$
$$+ O(\varepsilon^3) \, , \qquad (3.11)$$

$$H_0^{FP}(\underline{\pi}, \underline{\Phi}) = -\int_{R^d \otimes R^d} d\underline{r} \, d\underline{r}' \Upsilon(\underline{r}') \pi(\underline{r}) \frac{\delta \Xi_0[v^*(\underline{r})]}{\delta v^*(\underline{r}')}$$
$$+ \int_{R^d} d\underline{r} \, \pi(\underline{r})^2 \Xi_2[v^*(\underline{r})] + O(\varepsilon^3) \, ,$$

where $\Upsilon(\underline{r}) = \Phi(\underline{r}) - v^*(\underline{r})$, $\forall \underline{r} \in \mathbb{R}^d$; $|\Upsilon(\underline{r})| \sim |\pi(\underline{r})| \le \varepsilon$, $\forall \underline{r} \in R^d$; and the analyticity of the functionals Ξ is assumed. We see that a sufficient condition for the matching of Eqs. (3.11) up to order ε^3 is that Eqs. (3.7) hold.

Notice that it can be shown that the stationary points of the Hamiltonian dynamics generated by Eqs. (3.10) are hyperbolic. However, all trajectories meeting the above boundary condition belong to the corresponding fixed point stable manifold. Therefore, the ε^3 corrections in Eqs. (3.11) (which are different in each case) remain small under the evolution. The neighborhood of the fixed point should be appropriately taken in order to guarantee that the corrections to the nonequilibrium potential induced by the ε^3 terms are smaller than, for instance, $V_{1,st}$. Let us assume that $\nu \ge 3$ is the first leading correction to the

common part of the Hamiltonian expansions (3.11). The corresponding nonequilibrium potentials generated by those Hamiltonians differ in $O(\Omega\varepsilon^\nu)$. We want the latter difference to be of order Ω^{-1}. Therefore we should take $\varepsilon\sim\Omega^{-2/\nu}$.

Theorem 3. Let any FPE (3.1) be such that its coefficients have the form

$$\Xi_0[\Phi(\underline{r})]=\tfrac{1}{2}\{D_-(\Phi(\underline{r}))-D_+(\Phi(\underline{r}))\}\ ,$$

$$\Xi_1[\Phi(\underline{r})]=0\ ,\qquad(3.12)$$

$$\Xi_2[\Phi(\underline{r})]=\frac{D_-(\Phi(\underline{r}))-D_+(\Phi(\underline{r}))}{2\ln\left[\dfrac{D_-(\Phi(r))}{D_+(\Phi(\underline{r}))}\right]}\ ,$$

where

$$D_\eta(\Phi(\underline{r}))=\langle\!\langle D(U_\eta[\Phi(\underline{r});\underline{K}])\rangle\!\rangle\ .\qquad(3.13)$$

Then for

$$f(\eta)=\tfrac{1}{2}[\delta(\eta-1)+\delta(\eta+1)]\ ,\qquad(3.14)$$

(i) It reproduces the exact deterministic dynamics given by Eqs. (2.12) and (2.13) and (ii) the $V_{0,\mathrm{st}}$ and $V_{1,\mathrm{st}}$ parts of its stationary solution almost coincide with the exact one in a suitable neighborhood of all spatially homogeneous deterministic solutions $\underline{v}^*$.

Proof. (i) Same as (i) in Theorem 1. (ii) The argument goes through several well defined steps. (a) Let us define the functional $\pi^0[\Phi(\underline{r})]$, which is the solution of the implicit equation

$$\int_R d\eta\, f(\eta)\langle\!\langle D(U_\eta[\Phi(\underline{r});\underline{K}])\rangle\!\rangle[e^{\eta\pi^0[\Phi(\underline{r})]}-1]=0\ .$$
$$(3.15)$$

For example, when $f(\eta)$ is given by Eq. (3.14), it takes the value $\pi^0[\Phi(\underline{r})]=\ln\{D_-(\Phi(\underline{r}))/D_+(\Phi(\underline{r}))\}$, where $D_\pm$ are given by Eq. (3.13). This functional has an interesting property: its zeros are solutions of the deter-

ministic equation. This property is easily shown.

Let $\underline{v}'$ a zero of $\underline{\pi}^0$, i.e., $\pi^0[\underline{v}'(\underline{r})]=0$. We expand Eq. (3.15) around this zero and by keeping the leading order on the perturbation expansion, we find that $\underline{v}'$ is equal to the zero of the right-hand side of the deterministic equation (2.13) and therefore it is a stationary solution of it. In fact, from Eq. (2.14), it is straightforward to see that all spatially homogeneous solutions of the deterministic equation should be a zero of π^0. Let us focus on those spatially homogeneous solutions $\underline{v}^*=\underline{v}_{\mathrm{hom}}$.

(b) Let us choose now the FPE (3.1) coefficients

$$\Xi_0[\Phi(\underline{r})]=\Xi_0^{\mathrm{exact}}[\Phi(\underline{r})]\ ,$$

$$\Xi_1[\Phi(\underline{r})]=0\ ,\qquad(3.16)$$

$$\Xi_2[\Phi(\underline{r})]=\Xi_0^{\mathrm{exact}}[\Phi(\underline{r})]/\pi^0[\Phi(\underline{r})]\ .$$

It is matter of simple algebra to show that the coefficient Ξ_2 can be expanded around $\underline{v}_{\mathrm{hom}}$ as

$$\Xi_2[\Phi(\underline{r})]=Z_2[\Phi(\underline{r})]+\frac{Z_3[\Phi(\underline{r})]}{Z_2[\Phi(\underline{r})]}\Xi_0[\Phi(\underline{r})]+O(\Xi_0^2)\ ,$$
$$(3.17)$$

where

$$Z_n=1/n!\int_R d\eta\, f(\eta)\eta^n\langle\!\langle D(U_\eta[\underline{v}_{\mathrm{hom}};\underline{K}])\rangle\!\rangle\ .$$

Therefore, we can apply Theorem 2 in order to conclude that the FPE defined by those coefficients almost reproduces $V_{0,\mathrm{st}}$ in the neighborhood of $\underline{v}_{\mathrm{hom}}$.

(c) In order to study $V_{1,\mathrm{st}}$ we should follow a similar reasoning as in Theorem 2. Let us assume that the components $V_{0,\mathrm{st}}^{\mathrm{exact}}$ and $V_{0,\mathrm{st}}^{\mathrm{FP}}$ are known. Similarly to Eqs. (3.8) and (3.9), the corresponding Hamiltonians whose dynamics will define the potentials $V_{1,\mathrm{st}}^{\mathrm{exact}}$ and $V_{1,\mathrm{st}}^{\mathrm{FP}}$ are, respectively,

$$H_1^{\mathrm{exact}}(\underline{\pi},\underline{\Phi})=\int_R d\underline{r}\int_R d\eta\, f(\eta)\frac{\eta}{2}\left[\frac{\delta}{\delta\Phi(\underline{r})}\left[\langle\!\langle D(U_\eta[\Phi(\underline{r});\underline{K}])\rangle\!\rangle(1+e^{\eta[\delta V_{0,\mathrm{st}}/\delta\Phi(\underline{r})]})\right]\right.$$

$$\left.-2\langle\!\langle D(U_\eta[\Phi(\underline{r});\underline{K}])\rangle\!\rangle e^{\eta[\delta V_{0,\mathrm{st}}/\delta\Phi(\underline{r})]}\pi(\underline{r})\right]\ ,\qquad(3.18)$$

$$H_1^{\mathrm{FP}}(\underline{\pi},\underline{\Phi})=\int_R d\underline{r}\left[\frac{\delta\Xi_0[\Phi(\underline{r})]}{\delta\Phi(\underline{r})}-\frac{\delta\Xi_2}{\delta\Phi(\underline{r})}\frac{\delta V_{0,t}}{\delta\Phi(\underline{r})}-\Xi_2[\Phi(\underline{r})]\frac{\delta^2 V_{0,t}}{\delta\Phi(\underline{r})^2}+\left[2\Xi_2[\Phi(\underline{r})]\frac{\delta V_{0,t}}{\delta\Phi(\underline{r})}-\Xi_0[\Phi(\underline{r})]\right]\pi(\underline{r})\right]\ .$$

$$(3.19)$$

In contrast with the above Hamiltonians (3.8) and (3.9), now the fixed points for the corresponding Hamiltonian dynamical equations (3.10) have, in general, nonzero momenta $N=(\pi_{\mathrm{hom}},\underline{v}_{\mathrm{hom}})$ and therefore we should check first that both Hamiltonians lead to the same fixed point. This is done by setting the corresponding equations (3.10) equal to zero. It can be shown then that the sufficient

condition for both Hamiltonians to have the same fixed point is that $Z_3(\underline{v}_{\mathrm{hom}})=0$. This condition restricts us to the $f(\eta)$ given by Eq. (3.14).

Finally, it is simple algebra to expand the Hamiltonians $H_1^{\mathrm{exact,FP}}$ around the common fixed point and to check that up to second order in the parameter expansion both of them coincide and therefore they lead, locally, al-

most to the same potential $V_{1,\text{st}}$. Obviously, all the comments given in Theorem 2 apply here.

Corollary. The FPE defined in Theorem 3 is the same as the one in Theorem 1 when $p(\underline{K})=\delta(\underline{K}-\underline{K}_0)$ and $f(\eta)=\frac{1}{2}[\delta(\eta-1)+\delta(\eta+1)]$.

Proof. By direct application of Theorem 1.

IV. CONCLUSION

Finally, let us stress some of the points we have worked out in this paper. For a particularly type of model with competing microscopic dynamics, we have built a Fokker-Planck equation which guarantees *a priori* a series of properties of the unknown exact stationary nonequilibrium potential. In particular. (i) The FPE depends explicitly on the microscopic rates, (ii) the dynamics associated with the deterministic part of the equation is exactly described, (iii) its stationary distribution is the corresponding equilibrium Gibbsian one (independent on the microscopic rate used) when a unique microscopic mechanism acts, and (iv) in general, its stationary nonequilibrium distribution is locally exact around the extremals of the true one. These properties could be very valuable in order to extract reliable information from the mesoscopic description. For example, at present we are studying the influence of the microscopic dynamics in the critical phenomenon associated with a phase transition.

APPENDIX A: RELATION BETWEEN DETERMINISTIC EQUATION AND $V_{0,t}$

From Eq. (2.10) we assumed that for large enough Ω the probability distribution for a given fixed time t can be written as

$$P_1^\Omega(\underline{\Phi})=N_t^\Omega\exp[-\Omega V_{0,t}(\underline{\Phi})+O(\Omega^0)]\ , \quad \text{(A1)}$$

where N_t^Ω is the normalization of the probability. Because $V_{0,t}$ is bounded by below, it has at least one minimum in its domain of definition. Expanding the time nonequilibrium potential around one of the $V_{0,t}$ minima, we get

$$P_t^\Omega(\underline{\Phi})=N_t^\Omega\exp\left[-\Omega\int_{R^d\otimes R^d}d\underline{r}\,d\underline{r}'\,A_t(\underline{r},\underline{r}')\Upsilon_t(\underline{r})\Upsilon_t(\underline{r}')\right.$$
$$\left.+O(\Omega\Upsilon^3,\Omega^0\Upsilon)\right]\ , \quad \text{(A2)}$$

where $\Upsilon_t(\underline{r})=\Phi(\underline{r})-\Phi_t(\underline{r})^*$ and $\underline{\Phi}_t^*$ is one minimum of $V_{0,t}(\underline{\Phi})$, i.e., $\delta V_{0,t}(\underline{\Phi})/\delta\Phi(\underline{r})|_{\underline{\Phi}=\underline{\Phi}^*}=0$. (Note: It is expected that around its absolute minima the potential is differentiable.) We see from (A2) that for all the configurations such that $\lim_{\Omega\to\infty}\Omega^{1/2}|\Upsilon_t(\underline{r})|=\infty$, $\forall\underline{r}\in R^d$, its probability to occur decays exponentially in Ω and therefore they are not going to influence the system dynamical behavior when $\Omega\to\infty$. However, the rest of configurations are important in such a limit. Let us consider any configuration such that $|\Upsilon_t(\underline{r})|\leq\Omega^{-1/2}$, $\forall\underline{r}\in R^d$. The dynamics of the probability is given by solving Eq. (2.11) for the leading order in Ω. In our case

we know the analytical form of the probability by Eq. (A2) and the only unknown is Φ_t^*. We can get a time differential equation for Φ_t^* by substituting Eq. (A2) into (2.11), rescaling time by $\tau=\Omega^{-1}t$, and keeping the leading orders in Ω; thus

$$\int_{R^d\otimes R^d}d\underline{r}\,d\underline{r}'\,A_t(\underline{r},\underline{r}')\partial_\tau\Phi_\tau^*(\underline{r})\Upsilon_t(\underline{r}')$$
$$=\int_{R^d\otimes R^d}d\underline{r}\,d\underline{r}'\int_R d\eta f(\eta)\eta\langle\!\langle D(U_\eta[\underline{\Phi}_\tau^*(\underline{r});\underline{K}])\rangle\!\rangle$$
$$\times A_t(\underline{r},\underline{r}')\Upsilon_t(\underline{r}')\ ,$$
$$|\Upsilon_t(\underline{r})|\leq\Omega^{-1/2}\ ,\quad\forall\underline{r}\in R^d\ .\quad\text{(A3)}$$

By comparing Eqs. (2.13) and (A3), we get that $\underline{\Phi}_t^*=\underline{v}_t$. Therefore, for a given initial nonequilibrium potential $V_{0,t}$, the time evolution of its absolute minimum $\underline{\Phi}_\tau^*$, $\tau\geq t$, is given by the solution of the deterministic equation (2.13) with $\underline{\Phi}_t^*$ as the initial datum, when $\Omega\to\infty$.

APPENDIX B: SOLUTIONS TO A HAMILTON-JACOBI EQUATION (REF. [26])

Let $S_t(\underline{v})$ be a defined functional and C^∞ on an open set of the configurational space $U\subset G(\underline{v})$ and on a time interval $[0,t']$. Let $R_t(\underline{\Phi},\underline{v})$ be a real valued C^∞ function on an open set of the phase space $V\subset F(\underline{\Phi},\underline{v})$ and on a time interval $[0,t']$. $R_t(\underline{\Phi},\underline{v})$ is such that there exists a nonsingular and invertible map ψ such that for each time $t\in[0,t']$, it transforms the point $(\underline{\Phi},\underline{v})\in V$ into the point $\psi(\underline{\Phi},\underline{v})=(\underline{\Phi}',\underline{v})$, with $\Phi'(\underline{r})=\delta R(\underline{\Phi},\underline{v})/\delta\underline{\Phi}(\underline{r})$. Let $S_t(\underline{v})$ be a solution of a Hamilton-Jacobi type of equation

$$\partial_t S_t(\underline{v})+R_t\left[\frac{\delta S_t(\underline{v})}{\delta\underline{v}},\underline{v}\right]=0\ . \quad \text{(B1)}$$

Let us consider the differential equation

$$\partial_t v_t(\underline{r})=\frac{\delta R_t\left[\dfrac{\delta S_t(\underline{v}_t)}{\delta\underline{v}_t},\underline{v}_t\right]}{\delta\dfrac{\delta S_t(\underline{v}_t)}{\delta v_t(\underline{r})}} \quad \text{(B2)}$$

with the initial condition $\underline{v}_{t_0}=\underline{v}^{(0)}$, $t_0\in[0,t']$. Let us assume that for all $\underline{v}^{(0)}\in U$ and $t_0\in[0,t']$, one can solve the differential equation (B2) for t near t_0.

Then, by setting

$$\pi_t(\underline{r})=\frac{\delta S_t(\underline{v}_t)}{\delta v_t(\underline{r})}\ , \quad \text{(B3)}$$

the functions $\underline{v}_t$ and $\underline{\pi}_t$ are solutions to the Hamilton equations corresponding to the Hamiltonian $R_t(\underline{\pi},\underline{v})$, verifying the initial date $\underline{v}_{t_0}=\underline{v}^{(0)}$ and $\pi_{t_0}(\underline{r})=\delta S_{t_0}(\underline{v}_{t_0})/\delta v_{t_0}(\underline{r})$. Finally,

$$S_t(\underline{v}_t)=\int_{t_0}^t d\tau[\underline{\pi}_\tau\partial_\tau\underline{v}_\tau-R_\tau(\underline{\pi}_\tau,\underline{v}_\tau)]\ . \quad \text{(B4)}$$

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